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Bifurcation Structures of Two-dimensional Poiseuille Flow

Makoto UMEKI

Department of Physics, University of Tokyo [1]

1 Introduction

Bifurcations of two-dimensional progressive waves (Tollmien-Schlichting waves) in plane Poiseuille flows are studied numerically for many sets of parameters , i.e. Reynolds number and the fundamental wavenumber. It is confirmed that Hopf bifurcations occur (at least) twice on TS wave and successive instabilities lead to chaotic states in all examined cases of the fundamental wavenumber, as Reynolds number increases. The global bifurcation diagram is made based on results of the numerical simulation.

2 Methods of numerical simulation

The method of numerical simulation is described in detail in Umeki (1993). The evolution equation for the disturbance of vorticity is solved in the spectral space. The nonlinear term is computed by the pseudo-spectral method. The second-order Crank-Nicolson scheme is invoked for the normal ($y-$) derivative in the viscous term and the second-order Adams-Bashforth method is used for the remaining terms except for the terms linear to the Fourier-Chebyschev coefficients of vorticity, which can be integrated exactly. We imposed the non-slip boundary condition at the wall and the constant-flux condition in order to determine the streamfunction (and the velocity) from the vorticity.

The number of modes in the streamwise ($x-$) and normal directions is chosed as $(N_x, N_y) = (16, 64)$. Since we use the two-third law in the pseudo-spectral method, the number of

[1]The affiliation during the period between 2 Aug. and 21 Oct. 1994 : Departamento de Física, Faculultad de Ciencias Físicas y Matemáticas, Universidad de Chile

effective modes is actually $(M_x, M_y) = (10, 42)$. (Here we define $M_x = [2N_x/3]$ and $M_y = [2(N_y + 1)/3]$. [] is the notation of Gauss.) The time step Δt is 1/60. Previously, the simulation for the case $(N_x, N_y) = (16, 32)$ was performed. Comparing with this, there exists a small shift of bifurcation curves although qualitative features are common.

The fortran program is coded so that cases of many pairs of parameters (Re : Reynolds number and α : the fundamental wavenumber) can be simulated in the single program. This enables us to make use of the ability of the vector supercomputer in the maximal way and to study the bifurcation problems of Navier-Stokes equation. The fundamental wavenumber α is varied from 0.8 to 1.7 with the step size $\Delta\alpha$=0.1 and Reynolds number is from 2000 to 12000, depending on α, with $\Delta Re = 300$ or 100.

In order to determine the range of Reynolds number before starting the numerical simulation, branches of TS waves are computed by sloving algebraic equations of a progressive solution in the spectral space with the reduced number of modes as (M_x, M_y)=(2,42). A three-dimensional view of the TS wave in the parameter space (α, Re) versus the energy integral of disturbance E_d is shown in Figure 1. (For the precise definition of E_d, see Umeki(1994).)

In the present simulation, the initial condition is chosen as

$$\hat{\psi}_{init} = 0.3(1 - y^2)^2 \cos \alpha x + 0.01(1 - y^2)^2 \cos 2\alpha x$$

so that we can observe growth or decay of the component of symmetry different from TS waves.

3 Results

Figure 2 shows power spectra of (x-directional) averages of vorticity at the upper wall (denoted by $< \omega >_{y=1}$). The sequence is visible from the first Hopf bifurcation to the chaotic state. In this paper we use a term "periodic state" for the time-periodic modulation of TS wave. Then the quantity $< \omega >_{y=1}$ is constant for the progressive TS wave, and periodic for the periodic states. It should be noted that the quantity (e.g. vorticity) *at one point* is time-periodic for the progressive TS wave. Similarly, we use "quasi-periodic" and "choatic" for the quasi-periodic and chaotic modulations. Thus if the flow is doubly-periodic in $< \omega >_{y=1}$, it implies triply-periodic in a one-point quantity.

Figure 3 shows the bifurcation diagram in the parameter (Re, α) space obtained by the numerical simulation. There exists a relatively wide region of a stable TS wave from $\alpha=$ 1.1 to 1.6. In addition, a chaotic state does not appear for $\alpha \geq 1.5$ and Re ¡ 12000.

To see how the component different from TS wave increases as the flow becomes turbulent, we decompose the energy integral of disturbance into two integrals; one is the same as TS wave (denoted by E_d^A) and the other is different from it (E_d^B).

Figure 4 shows the temporal evolution of E_d^A and E_d^B for three cases (α, Re) =(1.1, 5000), (1.1, 6000) and (1.1, 9000), the former two of which correspond to periodic states and the last to a chaotic state. It is observed that E_d^B has a non-vanishing value, although it is very small, at the periodic state. It indicates that there is also a bifurcation from a state of symmetry with TS wave to a non-symmetric state in the periodic state. In the chaotic case of $Re = 9000$, the both components E_d^A and E_d^B are in the same order.

4 Conclusion

The previous studies [1–4)] on bifurcations in plane Poiseuille flows are extended to clarify the global bifurcation structure in the parameter space by means of numerical simulation. Bifurcation diagrams are expected to be produced for the three-dimensional flow or the subharmonic instability (i.e. the long channel [$\alpha \ll 1$]), which will be modified based on the diagram presented here.

5 References

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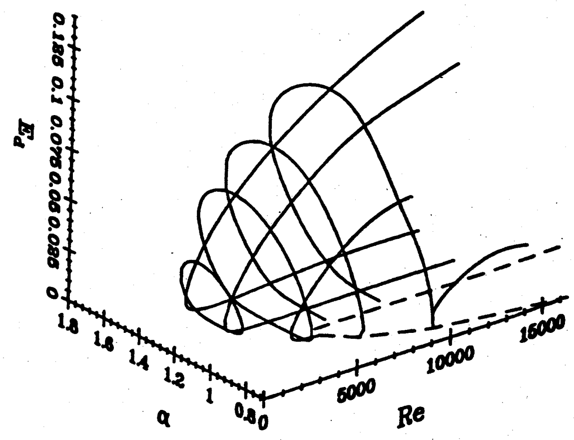

Figure 1. Three-dimensional view of TS wave of plane Poiseuille flow in the (α, Re, E_d) space. A dashed line denotes the neutral curve of the linear stability.

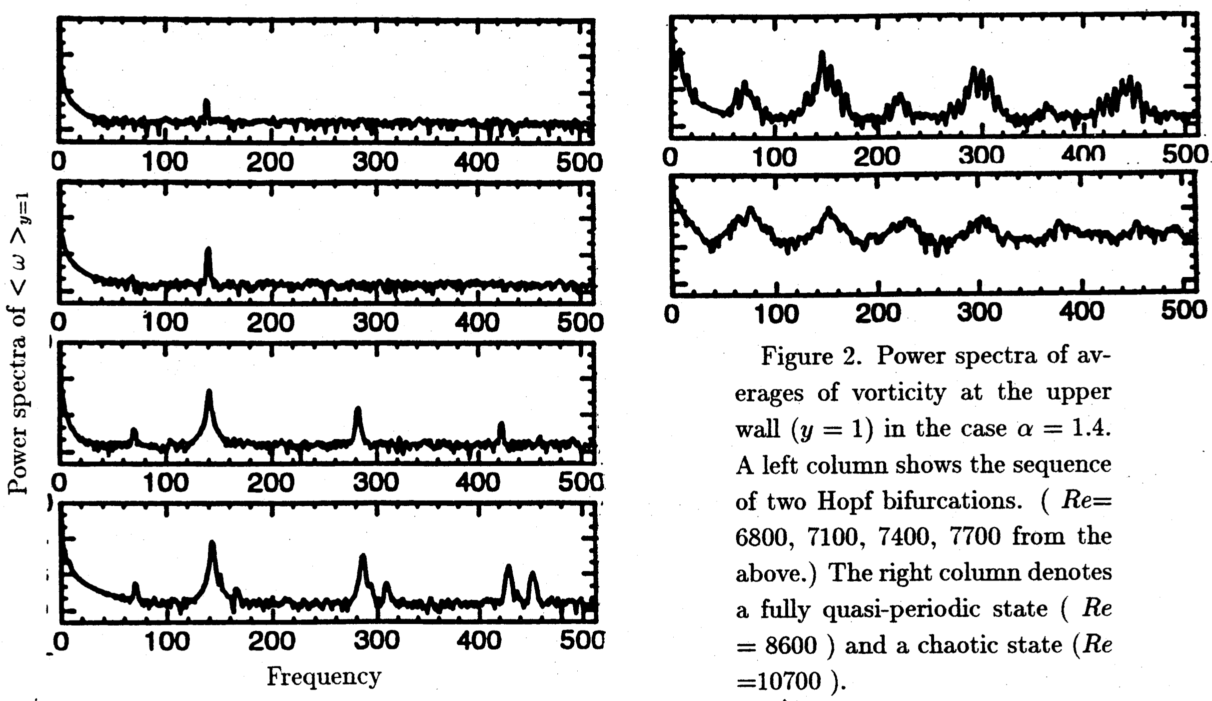

Figure 2. Power spectra of averages of vorticity at the upper wall ($y = 1$) in the case $\alpha = 1.4$. A left column shows the sequence of two Hopf bifurcations. (Re= 6800, 7100, 7400, 7700 from the above.) The right column denotes a fully quasi-periodic state (Re = 8600) and a chaotic state (Re =10700).

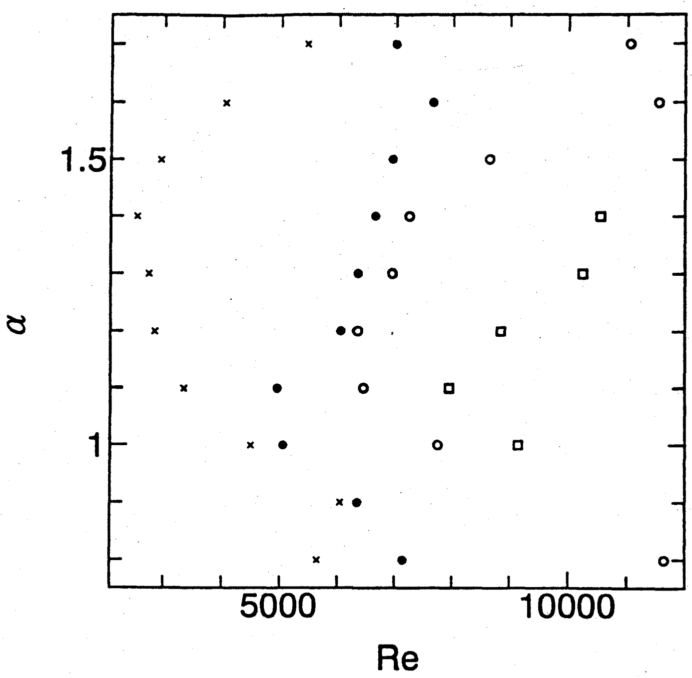

Figure 3. A global bifurcation diagram predicted by the numerical simulation. Boundaries are denoted by crosses (between laminar Poiseuille flow and TS wave), solid circles (TS wave and periodic state), open circles (periodic and quasi-periodic states), and squares (quasi-periodic and chaotic states).

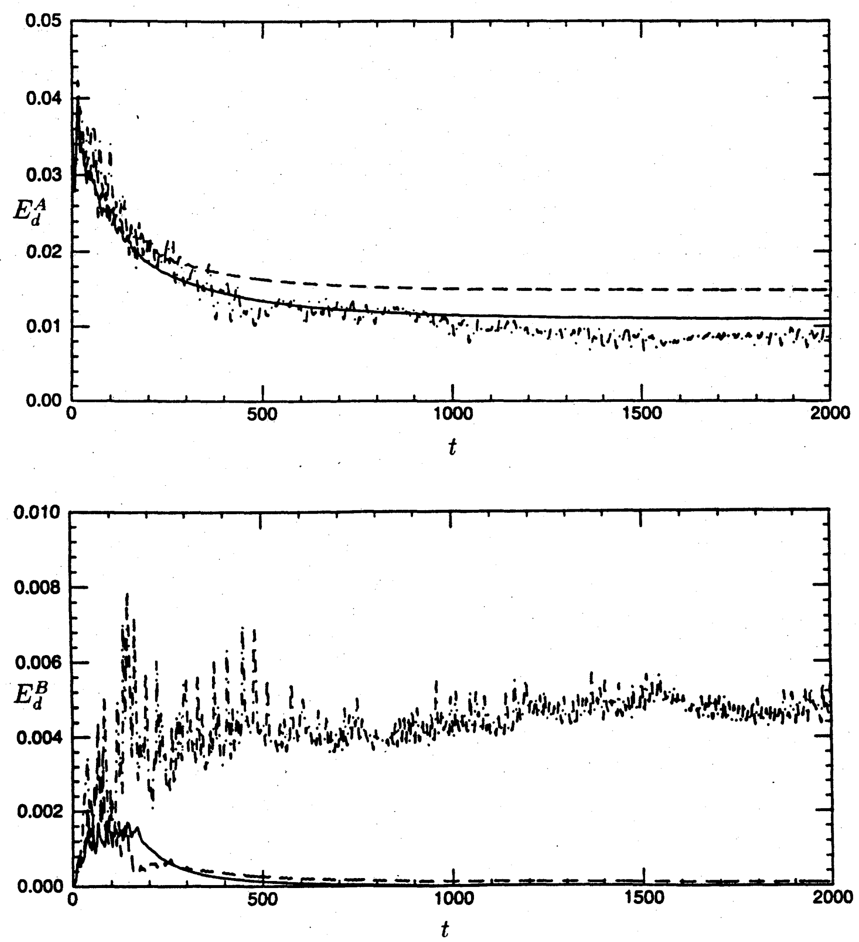

Figure 4. Temporal evolution of disturbance energy of symmetry which is the same as TS waves (the upper) and different from it (the lower). Solid, dashed and dotted curves denote respectively Re=5000, 6000 and 9000. α= 1.1.